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Temperature and measurement: comparison between two models of nonequilibrium radiation

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Abstract

We generalize a previous proposal of an experiment (J. Fort, D. Jou, J.E. Llebot, Physica A 248 (1998) 97) that, on the basis of predicted corrections to the Planck spectral law, may allow to determine experimentally what quantity is measured by a thermometer in steady-state nonequilibrium systems composed of matter and radiation. We include the photon number flux as an additional constraint. According to the model presented, the differences between the spectra predicted from two possible temperature definitions are higher than previously thought. In contrast to some previous papers, our model does not consider a specific matter content for the system. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

What is the correct definition of temperature in nonequilibrium systems? This is one of the most fundamental and controversial questions in nonequilibrium thermodynamics at present. In the last few years, the conceptual debate [1–6] has led to some specific approaches to the possibility of clarifying this question experimentally [7–10,35]. Such approaches rely on the fact that second-order nonequilibrium predictions depend on how the temperature is defined. These differences are expected to be important or not in practice depending on the phenomenon considered. For example, the differences between the predictions following from different temperature definitions in chemically

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reacting systems under heat conduction are expected to be negligible for readily attainable temperature gradients [11,36,37]. In contrast, much more relevant differences have been predicted in radiative transfer [10].

For classical monatomic gases *in equilibrium*, it is well-known that $\rho u_m = \frac{3}{2}nkT$, where ρ is the mass density, u_m the internal energy per unit mass, n the number density of molecules, k the Boltzmann constant and T is the temperature. From this *equilibrium* result, one might be tempted to expect that the quantity T_K , defined through

$$\rho u_m \equiv \frac{3}{2}nkT_K, \quad (1)$$

corresponds to the temperature in *nonequilibrium* systems as well. In fact, this position is often adopted in the kinetic theory of gases [12], as well as in most computer simulations. In order to generalize this hypothesis, one could then assume a similar nonequilibrium extension of the equilibrium results for other systems [12], i.e. $\rho u_m = \frac{l}{2}nkT_K$, with $l = 5$ for diatomic gases, $l = 6$ for solids, etc. However, in quantum systems equipartition no longer holds, thus this procedure cannot be applied to an arbitrary physical system. For example, for *equilibrium* radiation one has $\tilde{u}_r = aT^4$, with \tilde{u}_r the radiation energy density and $a \equiv 8\pi^5 k^4 / 15c^3 h^3$ (c is the speed of light *in vacuo* and h is the Planck constant). Of course one might assume [13] that T_r , defined through

$$\tilde{u}_r \equiv aT_r^4, \quad (2)$$

were the temperature of *nonequilibrium* radiation, but this completely different definition would certainly complicate the framework. If a simple, unifying theory of nonequilibrium systems exists, it seems that one single equation defining the temperature should hold for arbitrary physical systems. Moreover, the thermodynamical derivation [14] of the *equilibrium* law $\tilde{u}_r = aT^4$ is based on a completely different definition of temperature, namely,

$$\frac{1}{T} \equiv \frac{\partial S}{\partial U}, \quad (3)$$

where S is the entropy and U is the energy of the system. It is this definition that allowed Boltzmann to derive the Stefan law [14], in complete agreement with experimental results. Again, it is this definition that unifies the concept of temperature in *equilibrium* classical and quantum systems. It is therefore appealing to approach to the concept of temperature in *nonequilibrium* systems by introducing the temperature as

$$\frac{1}{T} \equiv \frac{\partial s}{\partial u}, \quad (4)$$

where s is the entropy per unit mass and u is the energy per unit mass. It has not yet been concluded from experiment whether the definition (4) [1,2,10] corresponds to physical reality or not, i.e. whether a thermometer placed in a system out of equilibrium measures the quantity T defined by Eq. (4) or not. This is why it is important to analyze under what experimental conditions it might be possible to test the proposal (4).

It is worth to stress that s in Eq. (4) is not the local-equilibrium entropy per unit mass, unless the system is in local thermodynamic equilibrium. Therefore, in general T differs from the local-equilibrium temperature. Although many authors assume that the local-equilibrium temperature is the quantity measured by a thermometer, Luzzi and co-workers have presented indications [9] that the measured temperature is not the local-equilibrium temperature, but rather that given by Eq. (4). The derivative in Eq. (4) is carried out under some constant parameters (in equilibrium matter systems, these parameters are ρ and the concentrations; outside equilibrium, it is not known for sure what additional parameters should be selected).

Out of equilibrium, thermometers sensitive to different degrees of freedom could read different values for the temperature. For instance, some authors have proposed a temperature $T(\lambda)$ dependent on the wavelength of radiation [15,16]. On the other hand, a thermometer with perfectly black walls would in general read a temperature different to that read by a thermometer with perfectly reflecting walls. Both temperatures will, still, be different to that read by a thermometer with a vacuum chamber between its external walls and the mercury. The first thermometer would be sensible to the thermal state of both matter and radiation, the second one only to that of matter, and the third one only to radiation. Our approach is based on the analysis of the intensity of radiation, which can be measured by means of a spectrophotometer. In contrast to, e.g., a black thermometer, a spectrophotometer is not sensible to the collisions of matter molecules but only to the photons, so that spectrophotometric measurements are related to the thermal properties of radiation and not to those of matter. Thus, since the radiation intensity is different if written in terms of T than in terms of T_r [10], it could in principle become possible to determine which of these quantities, if any, is the correct temperature by comparing experimental spectra to theoretical ones (the theoretical ones should be computed by making use of the value for the temperature read independently by means of a thermometer sensible to the radiation part only, i.e. a thermometer of the third kind of those described above).

The problem we have summarized has been recently analyzed [10] by means of an information-statistical approach to radiative transfer, the starting point of which is the maximization of the entropy density of a system composed by matter and radiation and subject to a photon energy flux [17]. For the special case in which the matter part of the system is a classical ideal gas, the entropy density can be written as [16]

$$\rho s = \rho s_m + \rho s_r = -k \int_{R^3} \frac{d^3 p_m}{h^3} f_m \ln f_m + 2k \int_{R^3} \frac{d^3 p_r}{h^3} [(1 + f_r) \ln(1 + f_r) - f_r \ln f_r], \quad (5)$$

where the subindexes m and r stand for matter and radiation; s , p and f are the corresponding specific entropies, momenta and momentum distribution functions, respectively, and ρ is the matter density. The results in Refs. [17,10] rely on the maximization of the entropy density (5) under the constraints of fixed energy density,

molecular number density n and radiative heat flux \mathbf{F} , namely

$$\rho u = \rho u_m + \rho u_r = k \int_{R^3} \frac{d^3 p_m}{h^3} \frac{p_m^2}{2m} f_m + 2 \int_{R^3} \frac{d^3 p_r}{h^3} p_r c f_r, \quad (6)$$

$$n = \int_{R^3} \frac{d^3 p_m}{h^3} f_m, \quad (7)$$

$$\mathbf{F} = 2 \int_{R^3} \frac{d^3 p_r}{h^3} p_r c \mathbf{c} f_r, \quad (8)$$

where u_m is the specific internal energy of the matter part, u_r is the radiation energy (also per unit mass of matter), m is the molecular mass and c is the photon velocity. As it was stressed in Refs. [10,17], the former equations hold only for the special case in which the molecules are nonrelativistic and monatomic, and heat conduction and convection are negligible. One of the purposes of the present paper is to generalize this too restrictive assumption.

The radiation distribution derived in the way we have summarized had been obtained previously [18,19,38] and has been related to thermodynamical quantities in Ref. [17]. It has also been analyzed in Ref. [20], where it has been shown that such a distribution can be transformed into an equilibrium (i.e., Planckian) distribution by performing a Lorentz boost. The authors of Ref. [20] thus argued that such an approach does not describe a nonequilibrium system but an equilibrium one as seen by an observer moving relative to it. They proposed to maximize the entropy density under an additional constraint of vanishing photon number flux $\mathbf{J}_N = 0$, where

$$\mathbf{J}_N = 2 \int_{R^3} \frac{d^3 p_r}{h^3} \mathbf{c} f_r. \quad (9)$$

The possibility of requiring that $\mathbf{J}_N = 0$, proposed in Ref. [20], has been analyzed recently, and it has been pointed out [21] that it does not seem clear that one can conclude that a system is in equilibrium just because its intensity becomes Planckian in a specific frame. It can also be shown that the requirement $\mathbf{J}_N = 0$ yields an expression for the entropy production that is not semi-positive definite, which is at odds with the second law of thermodynamics [21]. In view of such considerations, it seems sensible to analyze the consequences of taking into account the constraint (9) on the number flux, but without imposing that $\mathbf{J}_N = 0$. This approach will be followed in the present paper. Moreover, in view of the problems we have recalled in the first paragraph, it is necessary to determine whether the constraint (9) is consistent or not with the claim [10] that it is possible in principle to determine experimentally what the correct definition of nonequilibrium temperature in radiative systems is. If the derivative in Eq. (4) is taken under a constant energy flux \mathbf{F} , we have one possible definition of temperature. If it is taken under constant \mathbf{F} and \mathbf{J}_N , we have another possible definition. We are interested in analyzing the consequences of the difference between those two temperatures, as well as of the fact that both of them are different from the local-equilibrium temperature. This is the central problem that we will tackle here, and

it exemplifies very clearly the well-known fact [22,23,39] that in information theory there is not a well-established, general criterium to select the constrains under which the entropy density should be maximized.

The plan of the paper is as follows. In Section 2 we consider the nonequilibrium radiation intensity by taking into account the additional constraint (9), and compare it to previous work in which this constraint was not included. Section 3 is devoted to an analysis of the consequences of different temperature definitions in the system considered. Section 4 includes some concluding remarks.

2. The radiation distribution

The information-theoretical analysis of a matter–radiation system under the constraints (6)–(9) is included in Appendix A, where it is shown that, for the simple case in which the radiative heat flux \mathbf{F} is parallel to the x -axis,

$$\begin{aligned}
 f_r = & \frac{1}{\exp[\chi] - 1} \left(1 + \frac{\exp[\chi]}{\exp[\chi] - 1} \left\{ \left(p_r c^2 - \frac{18\zeta(3)kTc}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\} \cos \theta \right. \\
 & + \frac{\exp[\chi] (\exp[\chi] + 1)}{2 (\exp[\chi] - 1)^2} \left\{ \left(p_r c^2 - \frac{18\zeta(3)kTc}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\}^2 \cos^2 \theta \Bigg) \\
 & + O(3), \tag{10}
 \end{aligned}$$

where $O(3)$ stands for third- and higher-order terms, $\zeta(z)$ is the Riemann Zeta function, $\gamma = (\gamma, 0, 0)$ is the Lagrange multiplier concomitant to the radiative heat flux, θ is the angle between the x -axis and the photon velocity \mathbf{c} , and

$$\chi \equiv \frac{p_r c}{kT} \equiv \frac{p_r c}{k} \frac{\partial s}{\partial u},$$

so that the definition (4) of temperature has been applied. As it is shown in detail in Appendix A, the result (10) is valid provided that the microscopic formula for the entropy density of the matter content of the system can be written in a rather general form. In contrast, the distribution in Ref. [17] (which was applied in Ref. [10]) was derived for the special case of a nonrelativistic classical monatomic ideal gas. Moreover, the constraint (9) (which was not included in Refs. [10,17]) has been here taken into account.

In order to explore any experimentally testable consequence of the distribution (10), it is necessary to write down γ and J_N in terms of directly measurable quantities, such as the temperature and its gradient. In order to do so, we will first calculate the necessary components of the pressure tensor of radiation, which are given by [24,25]

$$P_{jx} = 2 \int_{R^3} \frac{d^3 p_r}{h^3} \frac{p_r}{c} c_j c_x f_r, \tag{11}$$

where $j = x, y, z$. Insertion of Eq. (10) and integration yields

$$P_{jx} = \frac{aT^4}{3} [1 + O(2)], \tag{12}$$

where use has been made of formula (3.411.1) in Ref. [26], and $O(2)$ stands for second- and higher-order terms (i.e., terms in $\gamma^2, J_N^2, \gamma J_N, \gamma^3$, etc.). It is not difficult to compute such terms explicitly after relatively tedious calculations but, as we shall see, in fact it is not necessary to do so for the purposes of the present paper. We have for the radiative energy flux (see Eq. (A.11) in Appendix A)

$$F = 4 \left(\frac{1}{3} - \frac{135}{\pi^6} [\zeta(3)]^2 \right) ac^2 k T^5 \gamma + \frac{18 \zeta(3)}{\pi^2} k T J_N + O(3). \tag{13}$$

The steady-state grey radiative transfer equation (RTE) reads [24,25]

$$\frac{c}{c} \cdot \nabla I_v = -\sigma I_v + j_v,$$

where σ is the absorption coefficient and j_v is the spectral emissivity.

Multiplication of the RTE by c/c and integration yields, after use of $I_v = 2h\nu^3 f_r/c^2$ [29] and of the definitions (8) and (11), the well-known result [27,28]

$$c \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} + \frac{\partial P_{zx}}{\partial z} \right) = -\sigma F. \tag{14}$$

Insertion of Eqs. (12) and (13) will yield an equation with two unknowns, namely γ and J_N . In order to derive another equation with the same unknowns, we multiply the RTE by $c/p_r c^2$, integrate and use the definitions (9) and $Q_{jx} = 2 \int_{R^3} \frac{d^3 p_r}{h^3} \frac{c_j c_x}{c^2} f_r$. This yields

$$c \left(\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yx}}{\partial y} + \frac{\partial Q_{zx}}{\partial z} \right) = -\sigma J_N. \tag{15}$$

We insert Eqs. (12) and (13) in Eq. (14). Similarly, we use Eq. (10) and integrate to find

$$Q_{jx} = \frac{16\pi\zeta(3)k^3 T^3}{h^3 c^3} + O(2),$$

which we insert in Eq. (15). This yields a system of two equations with two unknowns. We assume for simplicity that the temperature gradient is uniform and find, after neglecting third- and higher-order terms in the usual way [17],

$$\gamma = -\frac{1}{1 - \frac{405}{\pi^6} [\zeta(3)]^2} \frac{1}{\sigma c k T^2} \left(\frac{dT}{dx} + \frac{27\zeta(3)\sigma k}{2\pi^2 ac T^2} J_N \right), \tag{16}$$

$$J_N = -\frac{48\pi\zeta(3)k^3 T^2}{h^3 c^2 \sigma} \frac{dT}{dx}, \tag{17}$$

which relate γ and J_N to measurable quantities and are valid up to second order. Notice that second-order terms in, e.g., Eq. (12), give rise to third-order terms in Eq. (14), and are therefore negligible in the order of approximation considered. The energy flux can be related to the temperature gradient by means of Eqs. (13) and (16),

$$F = -\frac{4}{3} \frac{ac T^3}{\sigma} \frac{dT}{dx} + O(3).$$

It is worth to mention that this relationship is the same as that obtained in the model in which the additional constraint of fixed J_N was not included in the entropy maximization [17]. On the other hand, it is simple to estimate of the importance of the fluxes by comparing them to their maximum possible values,

$$\frac{|J_N|}{n_r c} = \frac{3}{\sigma T} \left| \frac{dT}{dx} \right| + O(2), \quad \frac{|F|}{\tilde{u}_r c} = \frac{4}{3\sigma T} \left| \frac{dT}{dx} \right| + O(2),$$

where $n_r = 2 \int_{R^3} \frac{d^3 p_r}{h^3} f_r$ is the photon number density. The denominators in the left-hand side of these two equations correspond to the extreme anisotropic limit in which all photons would move in the same direction. For the values $\sigma = 0.1 \text{ m}^{-1}$ [17], $T = 2000 \text{ K}$ and a temperature gradient of $dT/dx = 15 \text{ K/m}$, we find that J_N and F have about 23% and 10% of their maximum possible values, respectively. This estimation, in itself, provides an indication that it is relevant to incorporate the flux J_N to the description of the system, as done in the present paper. As explained below, this conclusion can also be reached from the analysis of the predicted spectra.

The intensity of radiation is related to the photon distribution function through $I_\lambda = \frac{2hc^2}{\lambda^5} f_r$ [17,29], where λ is the wavelength. Thus, making use of Eq. (10) and of the fact that the energy of a photon is $p_r c = hc/\lambda$,

$$I_\lambda = I_{\lambda Planck} (1 + \phi^{(1)} \cos \theta + \phi^{(2)} \cos^2 \theta), \tag{18}$$

where $I_{\lambda Planck}$ is the Planck function, i.e.

$$I_{\lambda Planck} = \frac{2hc^2}{\lambda^5} \frac{1}{e^\chi - 1}, \tag{19}$$

$$\phi^{(1)} = - \left\{ \left(\chi - \frac{18\zeta(3)}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\} \frac{e^\chi}{e^\chi - 1}, \tag{20}$$

$$\phi^{(2)} = \left\{ \left(\chi - \frac{18\zeta(3)}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\}^2 \frac{e^\chi (e^\chi + 1)}{2(e^\chi - 1)^2}, \tag{21}$$

with

$$\chi = \frac{hc}{kT\lambda}. \tag{22}$$

It is worth to mention that if we set $J_N = 0$ we would recover the results [21] corresponding to the proposal in Ref. [20], but numerical estimations show that the differences between such results and those above are not negligible, i.e. that the terms in J_N , obtained above, should be included in general.

We consider a matter system under a temperature gradient (see Fig. 1), and study the radiation it emits. A spectrophotometer measures the intensity due to all photons that cross a unit area coming from all possible directions of a hemisphere. For example, the intensity due to the photons that leave the system through a unit area, orthogonal to the x -axis and centered at point B in Fig. 1 is

$$i_{\lambda B} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \sin \theta \cos \theta I_{\lambda B}, \tag{23}$$

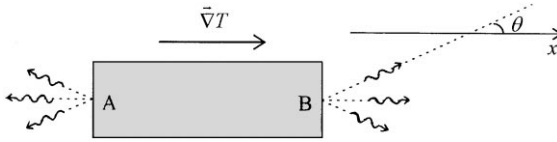


Fig. 1. A matter system with a nonuniform temperature distribution. This system generalizes a previously-proposed experiment (see Fig. 1 in Ref. [10]): the matter part of the system no longer needs to be assumed a classical non-relativistic monatomic ideal gas. In Figs. 2 and 3 we illustrate the predictions for the radiation emitted by the system depicted here, making use of two possible temperature definitions.

where $I_{\lambda B}$ is given by Eqs. (18)–(21) evaluated at point B, i.e. $T = T_B$ and $\frac{dT}{dx} = \frac{dT}{dx}|_B$. Similarly,

$$i_{\lambda A} = - \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} d\theta \sin \theta \cos \theta I_{\lambda A} . \tag{24}$$

Use of Eq. (18) into Eqs. (23) and (24) and integration yields

$$i_{\lambda A} = i_{\lambda Planck A} \left(1 - \frac{2}{3} \phi_A^{(1)} + \frac{1}{2} \phi_A^{(2)} \right) , \tag{25}$$

$$i_{\lambda B} = i_{\lambda Planck B} \left(1 + \frac{2}{3} \phi_B^{(1)} + \frac{1}{2} \phi_B^{(2)} \right) , \tag{26}$$

where the subindexes A and B refer to these points in Fig. 1, and

$$i_{\lambda Planck} = \pi I_{\lambda Planck} . \tag{27}$$

Let us assume, in order to illustrate the predictions of the model, that $\sigma = 0.1 \text{ m}^{-1}$ [17], $T_A = 2000 \text{ K}$, $T_B = 2001 \text{ K}$, and $dT/dx = 15 \text{ K/m}$. In Fig. 2 we plot the spectra for point A in Fig. 1, according to the Planckian or equilibrium result (27), and to the model considered in the present paper (both in the first-order approximation, i.e., $i_{\lambda A}^{(1)} = i_{\lambda Planck} (1 + \frac{2}{3} \phi_A^{(1)})$, and in the second-order approximation, Eq. (25)). For comparison purposes, Fig. 2 also includes the spectra predicted for the same values of the parameters but making use of the model obtained in Ref. [10]. That model makes use of the energy density and energy flux as constraints, but not of the additional constraint on the photon number flux. We see that the predictions are rather different depending on the model considered: for example, the model in Ref. [10] predicts (for wavelengths close to the intensity maxima) a correction of about 26% with respect to the Planckian intensity in the first approximation. By contrast, the model analyzed in the present paper predicts (in the same range) a correction of about 10% in the first approximation. Such differences between both models indicate that the question of the measurable temperature should be analyzed in detail for the model considered here, and this problem is tackled in the next section. Let us mention that it might at first sight seem that the areas under all of the curves in Fig. 2 should be the same, because we have calculated the intensities for a fixed energy density, energy flux, etc. However, the areas under these curves do not correspond to the energy density, neither to the energy flux. These areas correspond to part of the energy flux, specifically to the part

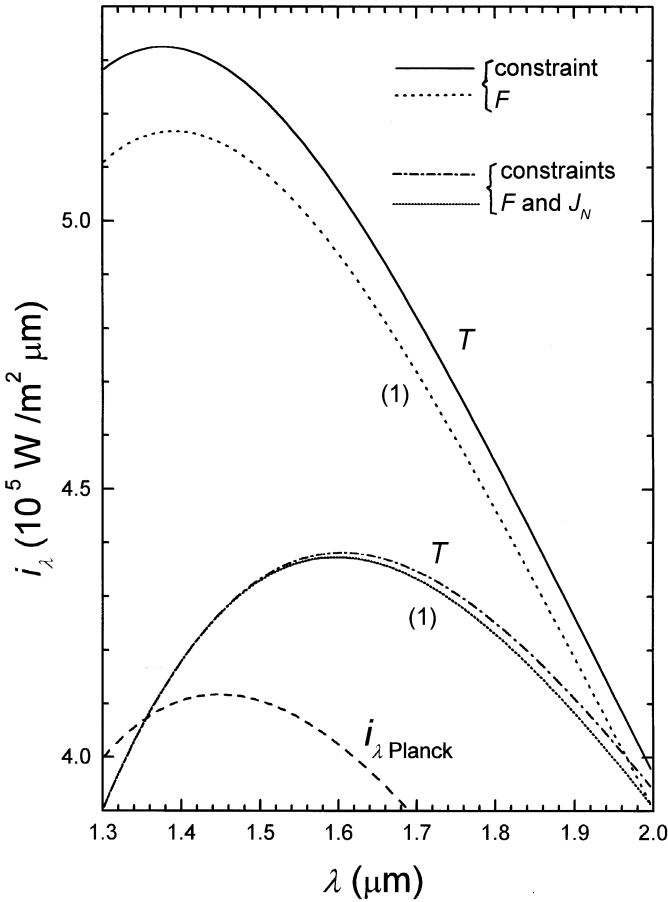


Fig. 2. Spectra for the radiation leaving the system in Fig. 1 through point A, according to the model in the present paper (constraints on F and J_N) and to that in Ref. [10] (constraint on F). The symbol (1) identifies first-order spectra. The symbol T identifies second-order spectra assuming T is the measurable temperature. $T_A = 2000$ K, $dT/dx = 15$ K/m, $\sigma = 0.1$ m $^{-1}$.

due to photons leaving the system in Fig. 1 across a unit area centered at point A, i.e. to photons moving to the left (photons with $c_x < 0$). This corresponds, in practice, to the fact that a spectrophotometer detects photons coming only from one side, whereas the energy flux is a net flux (photons with any possible value of c_x).

3. The role of temperature definitions in measurements of nonequilibrium radiation

In order to compare the implications of the temperature definitions (4) and (2) for the model considered in the former section, we must first relate the quantity T , defined by Eq. (4), to T_r , defined by Eq. (2). In order to do so, we substitute the distribution function (10) into $\tilde{u}_r = \rho u_r$, as given by Eq. (6) or (A.2). This yields, after performing

the integrals,

$$\tilde{u}_r = aT^4(1 + A_T\gamma^2 + B_TJ_N^2 + C_T\gamma J_N),$$

where use has been made of formula (3.411.1) in Ref. [26],

$$A_T \equiv \frac{10}{3} \left(1 - \frac{405}{\pi^6} [\zeta(3)]^2 \right) k^2 T^2 c^2,$$

$$B_T \equiv \frac{9\pi^2 k^2}{10a^2 c^2 T^6}, \quad C_T \equiv \frac{18\zeta(3)k^2}{\pi^2 a T^2},$$

and we have introduced the blackbody constant $a \equiv 8\pi^5 k^4 / 15h^3 c^3$. Comparing this result for \tilde{u}_r to Eq. (2) and neglecting higher-order terms, we can relate T to T_r ,

$$T = T_r \left(1 - \frac{A_{Tr}}{4} \gamma^2 - \frac{B_{Tr}}{4} J_N^2 - \frac{C_{Tr}}{4} \gamma J_N \right) + O(3), \tag{28}$$

where

$$A_{Tr} \equiv \frac{10}{3} \left(1 - \frac{405}{\pi^6} [\zeta(3)]^2 \right) k^2 T_r^2 c^2,$$

$$B_{Tr} \equiv \frac{9\pi^2 k^2}{10a^2 c^2 T_r^6}, \quad C_{Tr} \equiv \frac{18\zeta(3)k^2}{\pi^2 a T_r^2}.$$

For $\sigma = 0.1 \text{ m}^{-1}$, $T = 2000 \text{ K}$ and $dT/dx = 15 \text{ K/m}$, we obtain $T_r = 2033 \text{ K}$, i.e. the difference between T_r and T is of 1.6%. By contrast, for the model in which the role of J_N was not taken into account, this difference was less than 0.5% [10]. This makes us expect that the differences in measurable properties of the system (and specifically in the spectra) related to the quantity that one identifies with the temperature will be more important. In order to see this explicitly, let us note that we have found out the spectra, but so far only in terms of T and its gradient (see, e.g., Fig. 2). In order to write them in terms of T_r and its gradient, we make use of Eq. (28) into Eqs. (12) and (13). This yields, in the second-order approximation,

$$P_{jx} = \frac{aT_r^4}{3} [1 + O(2)], \tag{29}$$

$$F = 4 \left(\frac{1}{3} - \frac{135}{\pi^6} [\zeta(3)]^2 \right) ac^2 k T_r^5 \gamma + \frac{18\zeta(3)}{\pi^2} k T_r J_N + O(3). \tag{30}$$

We insert these results in Eq. (14), calculate $Q_{jx} = \frac{16\pi\zeta(3)k^3 T_r^3}{h^3 c^3} + O(2)$ and insert this in Eq. (15). We follow exactly the same method as in the previous section and find that

$$\gamma = - \frac{1}{1 - \frac{405}{\pi^6} [\zeta(3)]^2} \frac{1}{\sigma c k T_r^2} \left(\frac{dT_r}{dx} + \frac{27\zeta(3)\sigma k}{2\pi^2 a c T_r^2} J_N \right), \tag{31}$$

$$J_N = - \frac{48\pi\zeta(3)k^3 T_r^2}{h^3 c^2 \sigma} \frac{dT_r}{dx}, \tag{32}$$

which relate γ and J_N to T_r and $\frac{dT_r}{dx}$ (instead of T and $\frac{dT}{dx}$ (see Eqs. (16) and (17)). The energy flux is, from Eqs. (30) and (31),

$$F = -\frac{4}{3} \frac{acT_r^3}{\sigma} \frac{dT_r}{dx} + O(3).$$

Use of Eq. (28) into Eqs. (18)–(21) yields

$$I_\lambda = I_{\lambda r \text{ Planck}}(1 + \phi_r^{(1)} \cos \theta + \phi_r^{(2)} \cos^2 \theta + \phi_r^{*(2)}), \tag{33}$$

where

$$I_{\lambda r \text{ Planck}} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\lambda r} - 1}, \tag{34}$$

$$\phi^{(1)} = - \left\{ \left(\chi_r - \frac{18\zeta(3)}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T_r^3} J_N \right\} \frac{e^{\lambda r}}{e^{\lambda r} - 1}, \tag{35}$$

$$\phi^{(2)} = \left\{ \left(\chi_r - \frac{18\zeta(3)}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T_r^3} J_N \right\}^2 \frac{e^{\lambda r} (e^{\lambda r} + 1)}{2(e^{\lambda r} - 1)^2}, \tag{36}$$

$$\phi_r^{*(2)} = - \left(\frac{A_{Tr}}{4} \gamma^2 + \frac{B_{Tr}}{4} J_N^2 + \frac{C_{Tr}}{4} \gamma J_N \right) \chi_r \frac{e^{\lambda r}}{(e^{\lambda r} - 1)^2}, \tag{37}$$

$$\chi_r = \frac{hc}{kT_r \lambda}. \tag{38}$$

Finally, insertion of Eq. (33) into Eqs. (23) and (24) and integration yields

$$i_{\lambda r A} = i_{\lambda r \text{ Planck } A} \left(1 - \frac{2}{3} \phi_{rA}^{(1)} + \frac{1}{2} \phi_{rA}^{(2)} + \phi_{rA}^{*(2)} \right), \tag{39}$$

$$i_{\lambda r B} = i_{\lambda r \text{ Planck } B} \left(1 + \frac{2}{3} \phi_{rB}^{(1)} + \frac{1}{2} \phi_{rB}^{(2)} + \phi_{rB}^{*(2)} \right), \tag{40}$$

where

$$i_{\lambda r \text{ Planck}} = \pi I_{\lambda r \text{ Planck}}. \tag{41}$$

We can now address the question of whether the model presented predicts measurable differences between the cases in which T or T_r is assumed to correspond to the quantity measured by a thermometer. In order to do so, we plot in Fig. 3 the same spectra as those in Fig. 2 (i.e., $\sigma = 0.1 \text{ m}^{-1}$, $T_A = 2000 \text{ K}$, $T_B = 2001 \text{ K}$ and $dT/dx = 15 \text{ K/m}$) but include also the spectrum under the assumption that T_r is the measurable temperature (i.e., $\sigma = 0.1 \text{ m}^{-1}$ as before, but now $T_{rA} = 2000 \text{ K}$, $T_{rB} = 2001 \text{ K}$, and $dT_r/dx = 15 \text{ K/m}$).

4. Conclusions

The model analyzed here makes use of both fluxes F and J_N as constraints, in addition to the constraints (A.2) and (A.3). From Fig. 3 we find that this model predicts

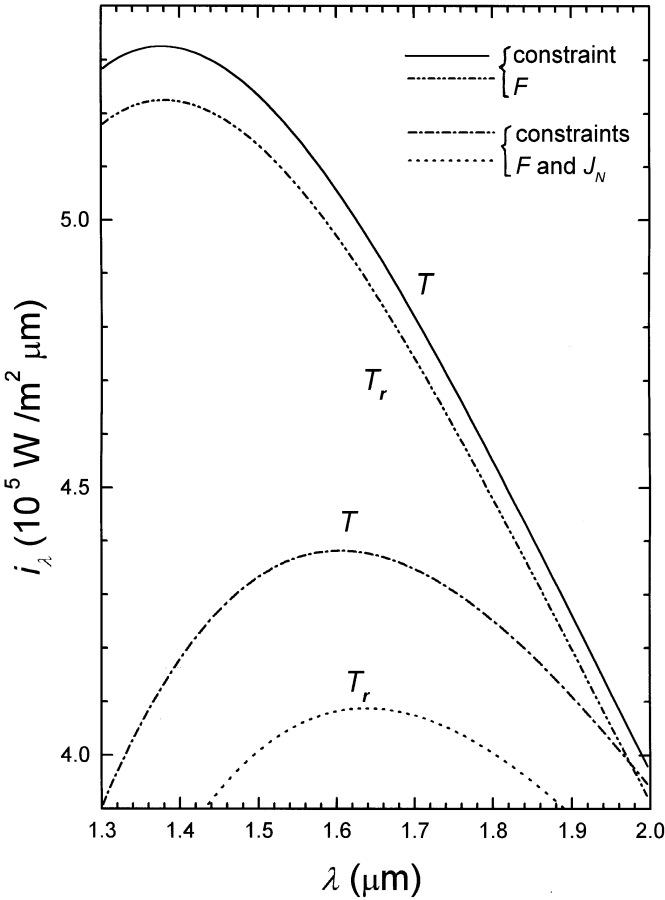


Fig. 3. The same as in Fig. 2, but here we plot second-order spectra. T and T_r identify the spectra obtained assuming T and T_r are the measurable temperature, respectively.

spectra that differ up to 8.5% depending on whether T (defined by Eq. (4)) or T_r (defined by Eq. (2)) is assumed to be measured by a thermometer. In contrast, and for exactly the same set of parameter values, the model which did not include the constraint on J_N predicted a difference of only 2% [10]. Thus the differences are rather higher than previously expected. We conclude that the inclusion of a constraint on the photon number flux confirms that it may be possible in principle to determine experimentally what quantity is measured by a thermometer. Moreover, since the measurable differences are higher in the present framework, the new model predicts that such an experimental approach should be, at least in principle, even simpler to carry out than previously thought. Although we have here dealt with the spectral differences, it is worth to mention that an additional way to tackle the problem may be based on the predictions for the wavelength of maximum intensity (i.e., the generalized Wien displacement) [30,21].

We have performed the same analysis for point B in Fig. 1 (instead of point A) and found similar results: this is why in Figs. 2 and 3 we have not included the spectra for point B but, instead, have preferred to present the spectra stemming from the previous model [10]. We stress that the differences between both models follow from the fact that we have here taken into account the requirement of the additional constraint on the photon number flux (Eq. (9)) in the entropy maximization.

It is worth to stress that, in contrast to previous derivations and proposals [10,17], which were restricted to a classical nonrelativistic monatomic ideal gas for the matter part of the system, the experiment proposed in the present paper, and all of the derivations presented, are not restricted to this condition. This is not only very appealing conceptually, but it might also make it simpler to perform the relevant experimental measurements in the future.

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Appendix A. Derivation of Eq. (10)

The entropy density (5) has been used in Refs. [10,17]. However, ρs_m in Eq. (5) is given by the classical ideal gas entropy density. Since this corresponds to a very special case, we here generalize Eq. (5) by

$$\rho s = \rho s_m + \rho s_r = -k \int_{R^3} \frac{d^3 p_m}{h^3} F(f_m) + 2k \int_{R^3} \frac{d^3 p_r}{h^3} [(1 + f_r) \ln(1 + f_r) - f_r \ln f_r], \tag{A.1}$$

where $F(f_m) = f_m \ln f_m$ for a classical ideal gas, $F(f_m) = -(1 + f_m) \ln(1 + f_m) + f_m \ln f_m$ for a phonon gas, etc. The case of a phonon gas may be important in practice because it might be simpler to carry out the experiment depicted in Fig. 1 by using a crystal instead of a gas: we think that in the latter case heat convection could complicate measurements. On the other hand, for nonideal systems the function $F(f_m)$ in Eq. (A.1) would also depend on the two-particle distribution function; for denser gases, the three-particle distribution function should also be taken into account, etc. [32]. For the sake of simplicity, such cases will not be considered here.

Similarly, we generalize Eqs. (6) and (7) by

$$\rho u = \rho u_m + \rho u_r = \int_{R^3} \frac{d^3 p_m}{h^3} H(p_m) f_m + 2 \int_{R^3} \frac{d^3 p_r}{h^3} p_r c f_r, \tag{A.2}$$

$$\langle A_l \rangle = \int_{R^3} \frac{d^3 p_m}{h^3} A^l(p_m) f_m, \tag{A.3}$$

with u the energy per unit mass and \mathbf{F} the radiative heat (or energy) flux. The microscopic operator $H(p_m)$ corresponds to the energy of the matter part (in the special case of nonrelativistic monatomic molecules we would have $H(p_m) = \frac{p_m^2}{2m}$, see Eq. (6); for a phonon gas $H(p_m) = \hbar\omega$, with ω the frequency of vibration and $\hbar = h/2\pi$; etc.). On the other hand, A^l are any additional operators (e.g., $n = \langle A_1 \rangle = \int_{R^3} \frac{d^3p_m}{h^3} f_m$, see Eq. (7)).

We make use of information statistical theory [31,33] by maximizing the entropy density (A.1) under the constraints (A.2), (A.3), (8) and (9), and finally obtain

$$\frac{d}{df_m} F(f_m) = -\beta H(p_m) - \sum_{l=1}^N \lambda_l A^l(p_m), \tag{A.4}$$

$$f_r = \frac{1}{\exp[\beta p_r c - \gamma \cdot p_r c \mathbf{c} + \boldsymbol{\delta} \cdot \mathbf{c}] - 1}, \tag{A.5}$$

where β , λ_l , γ and $\boldsymbol{\delta}$ are Lagrange multipliers (we have chosen the negative sign in front of γ in order to make the notation here similar to that in Ref. [17]). Eq. (A.5) is in agreement with that previously derived for a purely radiation system [34]. Here, we are considering a system composed of matter in addition to radiation, and this will allow us to relate all of the Lagrange multipliers to thermodynamical quantities.

The differential of the matter entropy density can be obtained from Eqs. (A.1), (A.4), (A.2) and (A.3). That of the radiation entropy density, from Eqs. (A.1), (A.5), (A.2), (8) and (9). The procedure is the usual one (see Appendix A in Ref. [17]), so we reproduce the final result for the case considered here,

$$ds = \frac{d(\rho s_m + \rho s_r)}{\rho} = k\beta du + \frac{k}{\rho} \sum_{l=1}^N \lambda_l d\langle A_l \rangle - \frac{k}{\rho} \gamma \cdot d\mathbf{F} + \frac{k}{\rho} \boldsymbol{\delta} \cdot d\mathbf{J}_N. \tag{A.6}$$

where it has been assumed for simplicity that the matter is macroscopically at rest, so that the equation of continuity implies that the matter density does not change in time, i.e. the density differential vanishes, $d\rho = 0$. From Eq. (A.6) and the thermodynamical definition of the temperature T , namely $\frac{1}{T} \equiv \frac{\partial s}{\partial u}$, we find

$$\beta = \frac{1}{kT}. \tag{A.7}$$

By assuming that near-equilibrium states correspond to small values of the radiation multipliers γ and $\boldsymbol{\delta}$, a second-order Taylor expansion of the RHS of (A.5) and use of Eq. (A.7) yields

$$f_r = \frac{1}{\exp[\chi] - 1} \left(1 + \frac{\exp[\chi]}{\exp[\chi] - 1} \mathbf{c} \cdot (p_r c \boldsymbol{\gamma} - \boldsymbol{\delta}) + \frac{\exp[\chi](\exp[\chi] + 1)}{2(\exp[\chi] - 1)^2} [\mathbf{c} \cdot (p_r c \boldsymbol{\gamma} - \boldsymbol{\delta})]^2 \right) + O(3), \tag{A.8}$$

where $O(3)$ stands for third- and higher-order terms, and $\chi = p_r c/kT$. We insert Eq. (A.8) into the constraints (9) and (8) and integrate. This yields

$$\delta = \frac{18\zeta(3)}{\pi^2} kT\gamma - \frac{9ch^3}{8\pi^3 k^3 T^3} \mathbf{J}_N, \tag{A.9}$$

$$\mathbf{F} = 4 \left(\frac{1}{3} - \frac{135}{\pi^6} [\zeta(3)]^2 \right) ac^2 kT^5 \gamma + \frac{18\zeta(3)}{\pi^2} kT \mathbf{J}_N, \tag{A.10}$$

where $\zeta(z)$ is the Riemann Zeta function, and the integrals have been performed by making use of the formulae (3.423,2) and (9.542,1) in Ref. [26].

We assume for simplicity a situation in which the photon energy and number fluxes are parallel to x -axis, i.e.

$$\mathbf{F} = (F, 0, 0) = \left(4ac^2 kT^5 \gamma \left[\frac{1}{3} - \frac{135}{\pi^6} [\zeta(3)]^2 \right] + \frac{18\zeta(3)}{\pi^2} kT J_N, 0, 0 \right), \tag{A.11}$$

where $\gamma = (\gamma, 0, 0)$ and $\mathbf{J}_N = (J_N, 0, 0)$. Making use of Eqs. (A.8) and (A.9) we have

$$\begin{aligned} f_r = & \frac{1}{\exp[\chi] - 1} \left(1 + \frac{\exp[\chi]}{\exp[\chi] - 1} \left\{ \left(p_r c^2 - \frac{18\zeta(3)kTc}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\} \cos \theta \right. \\ & \left. + \frac{\exp[\chi](\exp[\chi] + 1)}{2(\exp[\chi] - 1)^2} \left\{ \left(p_r c^2 - \frac{18\zeta(3)kTc}{\pi^2} \right) \gamma + \frac{9c^2 h^3}{8\pi^3 k^3 T^3} J_N \right\}^2 \cos^2 \theta \right) \\ & + O(3), \end{aligned} \tag{A.12}$$

where θ is the angle between the x -axis and the photon velocity \mathbf{c} . This completes the derivation of Eq. (10).

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