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On Approximate Solutions to the Wavefront Speed Problem

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We propose an approximate method to obtain the speed of wavefronts. It is built up from a known variational principle. For a range of systems of biological and physical interest, comparison to previously-known solutions and to numerical simulations shows the powerfulness of our approximate technique. For time-delayed equations, we also propose an alternative approximate solution, based on the renormalization group approach, and we compare both approximations.

KEY WORDS: Wavefronts; speed selection; variational principles; time-delayed diffusion.

1. INTRODUCTION

Reaction-diffusion wavefronts are ubiquitous in nature. One usually measures their asymptotic propagation speed and compares this experimental value to that predicted theoretically. This comparison between theory and experiment is performed not only in many physical contexts such as combustion fronts[(1)] but also in biological applications such as epidemics spread[(2)] and population invasions.[(3)] In Section 2, we present a detailed explanation of an approximate approach which we propose for this front speed problem. In Section 3, we compare the results of our method to some known solutions for the speed of fronts. Finally, Section 4 is devoted to wavefronts without analytically known speed, for which we compare our approximate analytical method to numerical simulations. For time-delayed

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diffusion, in Section 4 we also introduce another, alternative approximate solution, and we compare both of our approximations to the results of the numerical simulations.

2. FRONT SPEED PROBLEM

Consider a typical reaction-diffusion equation of the form

$$\partial_t \rho = \partial_{xx} \rho + f(\rho) \tag{1}$$

where $\rho(x, t)$ is a time-dependent field (particle number density, charge density, temperature, etc.), usually defined such that $\rho = 0$ and $\rho = 1$ correspond to an unstable and stable roots of the nonlinear function $f(\rho)$, respectively. By using the appropriate temporal and spatial variables (t and x respectively), Eq. (1) is used very often to describe the evolution of mass, energy or electric charge, which diffuses through the system under consideration (first term in the right-hand side) and is also produced due to a source process such as a chemical reaction or the reproduction of a living species (second term in the right-hand side). Let us define $z \equiv x - ct$, where $c > 0$ is the speed of solutions $\rho(x, t)$ that depend on x and t through z only, and such that they connect the two steady states with $\lim_{z \to -\infty} \rho = 1$ and $\lim_{z \to \infty} \rho = 0$. Then, from Eq. (1) we obtain a differential equation without partial derivatives,

$$\rho_{zz} + c\rho_z + f(\rho) = 0 \tag{2}$$

the solution of which is known to satisfy the Benguria–Depassier principle, namely[4]

$$c = \max_g \left[2 \frac{\int_0^1 \sqrt{fgh}\, d\rho}{\int_0^1 g\, d\rho} \right] \tag{3}$$

where $g(z) > 0$ satisfies the inequality $h \equiv -dg/d\rho > 0$, and moreover[4]

$$f \frac{g}{p} = hp \tag{4}$$

with $p \equiv -\rho_z$. As it stands, Eq. (3) is not enough in general to determine c with precision, because we do not know the function $g(z)$ such that the maximum is attained in the right hand side of Eq. (3). Therefore, we look for an approximate way to determine such a function $g(z)$ as follows. Since $\lim_{z \to -\infty} \rho = 1$ and $\lim_{z \to \infty} \rho = 0$, we may assume that $p > 0$ for all values

of z, and that ρ is a slowly varying function of z for high enough values of $|z|$, i.e.,

$$\rho_{zz} \ll \rho_z \tag{5}$$

thus Eq. (2) yields

$$c\rho_z + f(\rho) \simeq 0$$

and, from the definition of $p(z)$,

$$p \equiv -\rho_z \simeq f/c \tag{6}$$

This is not an exact result, so that we may introduce α such that

$$p \equiv f(\rho)/\alpha \tag{7}$$

On the other hand, from Eq. (4), $g = hp^2/f$ and since $h \equiv -dg/d\rho$, we may integrate and obtain

$$g(\rho) = \exp\left[-\int \frac{f}{p^2}\, d\rho \right] \tag{8}$$

For a given source function $f(\rho)$ in the reaction-diffusion equation (1), our method proceeds as follows. First, we obtain $g(\rho)$ from Eq. (8). In principle p is given by Eq. (7) with the exact source function $f(\rho)$ but in practice, we will use an approximate $f(\rho)$ in Eq. (7), such that it vanishes at $\rho = 0, 1$ and that the integral (8) can be performed analytically. After one has obtained an approximate $g(z)$ from Eq. (8), it will become possible to derive an approximate value for c by using Eqs. (3) and (4),

$$c = \max_{\alpha} \left[2 \frac{\int_0^1 (fg/p)\, d\rho}{\int_0^1 g\, d\rho} \right] \tag{9}$$

where the maximization is no longer over all arbitrary functions g (because an expression for g is now known from Eq. (8)), but over the parameter α, on which g depends through p (see Eq. (7)). Of course, α is such that the integrals in Eq. (9) exist.

We illustrate this procedure below. Before doing so, it is worth to note the following remarks.

(i) a multiplicative constant could have been included in Eq. (8), but according to Eq. (9) it would not affect the prediction for the wavefront speed c.

(ii) the exact function $p \equiv -\rho_z$ vanishes at $\rho \to 1$ and $\rho \to 0$. We will always find p by choosing an approximate source function f in (7) such that it vanishes at $\rho = 0, 1$: then, the leading contributions to the integral in the numerator of Eq. (9) come from $\rho \simeq 1$ and $\rho \simeq 0$, which are precisely the ranges where our approximation (6) holds.

3. COMPARISON TO EXACT SOLUTIONS FOR THE SPEED

For illustration purposes, in this section we briefly compare our analytical approach with some solutions for the speed which have appeared in the literature.

3.1. Cubic Model

The cubic model

$$f = \frac{\rho(1-\rho)(\rho+b)}{b} \tag{10}$$

has been used in genetics[5] and analyzed by Ben-Jacob.[6] In this case, after taking into account the remarks in the previous section, we see that we may use

$$p = f/\alpha \simeq \rho(1-\rho)/\alpha \tag{11}$$

and we obtain from Eq. (8) that

$$g = \rho^{-\alpha^2}(1-\rho)^{\alpha^2/b+\alpha^2} \tag{12}$$

Making use of Eqs. (10)–(12), we find

$$\int_0^1 g\, d\rho = \frac{\Gamma(1-\alpha^2)\,\Gamma(\alpha^2+\frac{\alpha^2}{b}+1)}{\Gamma(2+\frac{\alpha^2}{b})}$$

and

$$\int_0^1 \frac{fg}{p}\, d\rho = \frac{\alpha}{b}\left[\frac{\Gamma(2-\alpha^2)\,\Gamma(\alpha^2+\frac{\alpha^2}{b}+1)}{\Gamma(3+\frac{\alpha^2}{b})} + b\int_0^1 g\, d\rho\right]$$

where Γ is the gamma function, and the integrals have been solved with the use of formula (3.191) in ref. 7. These integrals exist if $\alpha < 1$.[7] From this, Eq. (9) and the well-known relationship $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$c \simeq 2(2b+1) \max_{0<\alpha<1} \alpha(2b+\alpha^2)^{-1}$$

which yields

$$c \simeq (2b)^{1/2} + (2b)^{-1/2} \tag{13}$$

The maximum is attained for $\alpha = \sqrt{2b} < 1$, so this result holds for $0 < b < 1/2$. Previous work derived Eq. (13) on the basis of an exact solution $\rho(z)$ to the reaction-diffusion equation (1). However, such exact profiles $\rho(z)$ do not determine all possible solutions and often do not give the most relevant one, since in general they do not travel with the speed truly selected by fronts (see ref. 10, p. 289, where this is explicitly shown for Fisher fronts). This is why Ben-Jacob and coauthors argued that Eq. (13) is a special value of the wavefront speed c.[6] In contrast, in our derivation above we have not needed a specific function $\rho(z)$, so that our method shows that Eq. (13) is the speed of fronts evolving according to Eqs. (1) and (10).

3.2. Bistable Systems

The so-called bistable systems are characterized by two stable states and an unstable one. The reaction term is given by

$$f = \rho(1-\rho)(\rho-a)$$

which has been applied to the study of nerve conduction.[8] In this case, there is an exact solution for Eq. (1) which has been derived in ref. 9. However, as mentioned above exact solutions do not include all possible solutions and thus do not allow to find the true speed selected by the front in general.[10] Let us derive the speed by means of our approximate method. Then, we do not need an exact solution for Eq. (1). We have

$$p = f/\alpha \simeq \rho(1-\rho)/\alpha$$

and (8) yields $g = \rho^{a\alpha^2}(1-\rho)^{(1-a)\alpha^2}$, which used into Eq. (9) leads us, after integration, to

$$c \simeq 2(1-2a) \max_{\alpha>0} \alpha(2+\alpha^2)^{-1} = \frac{1}{\sqrt{2}} - a\sqrt{2}$$

which coincides with the result in ref. 9. In the formalism presented in Section 2, we have assumed that a positive wavefront speed, $c > 0$.This will hold provided that $a < 1/2$.

3.3. Higher Power-Law Growth

The reaction term

$$f = \rho^{q+1}(1-\rho^q)$$

has a known analytical known solution with speed $c = 1/\sqrt{1+q}$.[10] Taking

$$p = f/\alpha \simeq \rho(1-\rho^q)/\alpha$$

and using the new variable $u \equiv \rho^q$, we can solve the integrals in Eq. (8) and obtain $g = (1-\rho^q)^{\alpha^2/q}$, which used into Eq. (9) yields

$$c \simeq \max_{\alpha > 0} 2\alpha/(1+q+\alpha^2) = 1/\sqrt{1+q}$$

where the maximum is reached for $\alpha = \sqrt{1+q}$, and the speed above coincides with that from the exact solution found in ref. 10.

3.4. The Ginzburg–Landau (G-L) Equation

G-L equations are very useful in the analysis of superconducting fronts.[11] The G-L equation

$$\phi_{zz} + c\phi_z + (1-\phi^2)(a+\phi) = 0$$

has been widely used in the literature and recently in ref. 12. In this equation the wavefront propagates into a metastable state and satisfies the boundary conditions $\phi(z \to \pm\infty) = \pm 1$. Calling $\rho = (1-\phi)/2$ one finds

$$\rho_{zz} + c\rho_z + 2\rho(1-\rho)(2\rho-(1+a)) = 0$$

where $\rho(z \to \pm\infty) = 0, 1$. In this case

$$p = f/\alpha \simeq \rho(1-\rho)/\alpha$$

and one obtains from Eq. (8) $g = \rho^{2\alpha^2(1+a)}(1-\rho)^{2\alpha^2(1-a)}$ and Eq. (9) yields

$$c \simeq \max_{\alpha} [-4a\alpha/(1+2\alpha^2)] = a\sqrt{2}$$

which agrees with the result given in ref. 12. The integrals in Eq. (9) exist for $\alpha^2 > a^2 - \frac{1}{2}$,[7] and the maximum is attained for $\alpha = -1/\sqrt{2}$. Combining both conditions, we find that the result obtained for the front speed c holds if $a^2 < 1/2$.

3.5. Schlogl's Second Model

This is a model for chemical reactions[13] and is given by

$$\phi_{zz} + c\phi_z + (1-\phi)(\phi-\phi_+)(\phi-\phi_-) = 0$$

where $\phi_\pm = 1 \pm \sqrt{1-\gamma}$ and $1 \leqslant \phi \leqslant \phi_+$. Calling $\rho = (\phi-1)/(1-\phi_+)$ one gets

$$\rho_{zz} + c\rho_z + (1-\gamma)\,\rho(1-\rho)(1+\rho) = 0$$

In this case,

$$p = f/\alpha \simeq \rho(1-\rho)/\alpha$$

and one obtains from Eq. (8) $g = \rho^{-\alpha^2(1-\gamma)}(1-\rho)^{2\alpha^2(1-\gamma)}$. After solving the integrals in Eq. (9), we find

$$c \simeq 6(1-\gamma) \max_\alpha \left[\alpha/\{2+\alpha^2(1-\gamma)\}\right]$$

$$= 3\sqrt{(1-\gamma)/2}$$

which is the exact result in ref. 13.

4. COMPARISON TO NUMERICAL SIMULATIONS

In the previous section, we have shown that our method makes it possible to derive some analytical results for the front speed in a variety of systems. In contrast to the usual methods, we have not used any exact solution for the front profile $\rho(z)$. We stress that this is important, because in general exact solutions do not have the true speed selected by the front.[10] We illustrate in this section how our method works on systems which have no known exact solutions, and we will also compare with the results for the front speed obtained from numerical integrations of the reaction-diffusion equation (1).

4.1. Forest Fires

In the case of chemical reactions of the form

$$nB+G \rightarrow (n+1)\,B \tag{14}$$

the source function for the population number density of species B ($\rho_B \equiv \rho$ in Eq. (1)) is

$$f(\rho) = \rho^n(1-\rho) \tag{15}$$

where we have considered that the total number density is a constant, which has been normalized ($\rho_B + \rho_G = 1$). The source term in (15) has been applied to the analysis of the spread of forest fires[14, 15] (then, B in Eq. (14) represents burning trees which set fire to green trees G). Since we are looking for solutions such that $\rho \rightarrow 1$ for $z \rightarrow -\infty$ and $\rho \rightarrow 0$ for $z \rightarrow \infty$, for large enough values of $|z|$ we see that $\rho(z)$ is a slowly varying function of z and

$$p \simeq f/\alpha = \rho^n(1-\rho)/\alpha \simeq \rho(1-\rho)/\alpha$$

This approximate trial function allows us to apply the Benguria–Depassier principle (3) as

$$c \simeq \max_{\alpha \in \mathscr{D}} \left(2\alpha \frac{\int_0^1 \rho^{n-1} g\, d\rho}{\int_0^1 g\, d\rho} \right) \tag{16}$$

where $\mathscr{D}$ is the set of positive values of α such that the integrals in (16) exist and g is given, from Eq. (8), by

$$g = \exp\left(-\alpha^2 \int \frac{\rho^{n-2}}{1-\rho}\, d\rho \right)$$

From this and Eq. (16) it is easy to find out c numerically for any value of the order parameter n in the kinetic mechanism (14). Previously, only the well-known cases $n=1$ and $n=2$ had been solved exactly,[10] whereas for $n>2$ only lower and upper bounds had been derived. The expressions for the lower and upper bounds are[16]

$$c_L = \max_{\alpha \in (0,\,1)} \frac{2\alpha \sqrt{1-\alpha}\, \Gamma(n/2+\alpha-1/2)\, \Gamma(3/2)}{\Gamma(n/2+\alpha+1)} \tag{17}$$

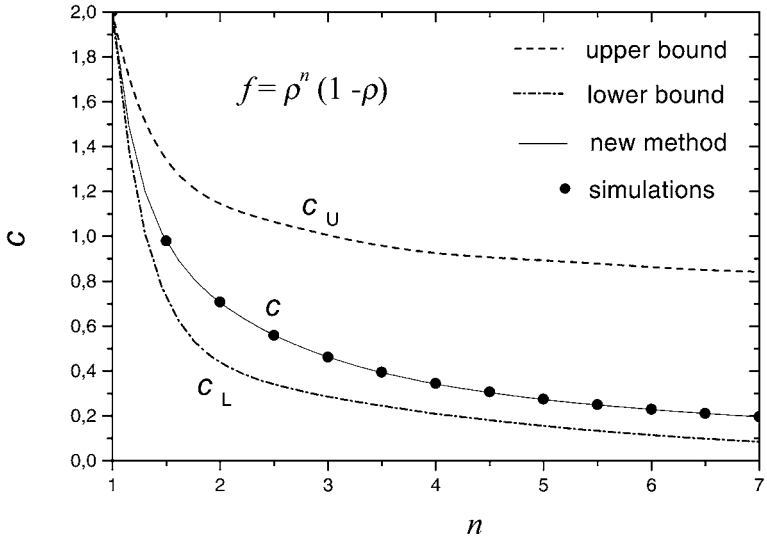

Fig. 1. Predictions for the speed of wavefront solutions to Eq. (1) for $f(\rho) = \rho^n(1-\rho)$. The dashed-dotted and dashed curves are lower and upper bounds, respectively, Eqs. (17)–(18), both of them predicted previously from the Benguria–Depassier approach.[16] The full curve is the prediction (16) derived in the text, and the black circles are the results of numerical simulations.

and

$$c_U = 2\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \tag{18}$$

In Fig. 1 we see that our new result (16) clearly improves these bounds, and gives very accurate values for the speed. Finally, we may note from Fig. 1 that c decreases for increasing values of n. This was to be expected intuitively since a higher value of n may be understood from the reaction scheme (14) as a higher number of burning trees required to set fire to a nearby green tree, which should reduce the propagation speed c of the fire.

4.2. Chemical Kinetics

Let us now study the model for chemical kinetics

$$nA + mB \to (n+m)\,A \tag{19}$$

The source function is in this case

$$f(\rho) = \rho^n(1-\rho)^m \tag{20}$$

which generalizes the case of forest fires, considered above, and has been applied to the spread of microorganisms.[17] For the sake of simplicity we study the case $n = 2$, that is, $f = \rho^2(1-\rho)^m$. Thus we have

$$p = f/\alpha \simeq \rho(1-\rho)/\alpha \tag{21}$$

From Eqs. (8) and (9) we find

$$g = \exp\left[\frac{\alpha^2}{m-1}(1-\rho)^{m-1}\right]$$

and

$$c \simeq \max_{\alpha} 2\left[\alpha \frac{\int_0^1 \rho(1-\rho)^{-1} g\, d\rho}{\int_0^1 g\, d\rho}\right] \tag{22}$$

As mentioned above, in previous papers the Benguria–Depassier principle had been used to derive lower and upper bounds for the speed. By following exactly the same procedure as in ref. 15, those bounds can be easily derived for the source function considered now ($f = \rho^2(1-\rho)^m$),

$$c_L = \max_{\alpha}\left[\frac{2\alpha\sqrt{1-\alpha}\,\Gamma(\alpha+1/2)\,\Gamma(m/2+1)}{\Gamma(m/2+\alpha+3/2)}\right]$$

$$c_U = 2[\sup_{\rho\in(0,1)}\{\rho(1-\rho)^{m-1}[2-(m+2)\rho]\}]^{1/2}$$

We use these bounds in order to illustrate, in Fig. 2, the differences between the results from the procedure in the present paper (which yields a specific, albeit approximate value for the speed, namely the full line in Fig. 2) and the approach in ref. 15 (which allows only to derive the lower and upper bounds [c_L and c_U above] for the speed, i.e., the broken curves in Fig. 2). From Fig. 2 we see that our new result (22) clearly improves the upper and lower bounds, and that good agreement with the results of numerical simulations is attained. The value of c decreases for increasing values of m, as was to be expected because more B molecules are needed in the chemical reaction (19), so that the reaction speed (20) diminishes for increasing values of m (recall that $0 < \rho < 1$, and that the same reason explains the effect of parameter n appearing in Eq. (15)).We may also note that our approximate result (full line in Fig. 2) is less precise the higher the

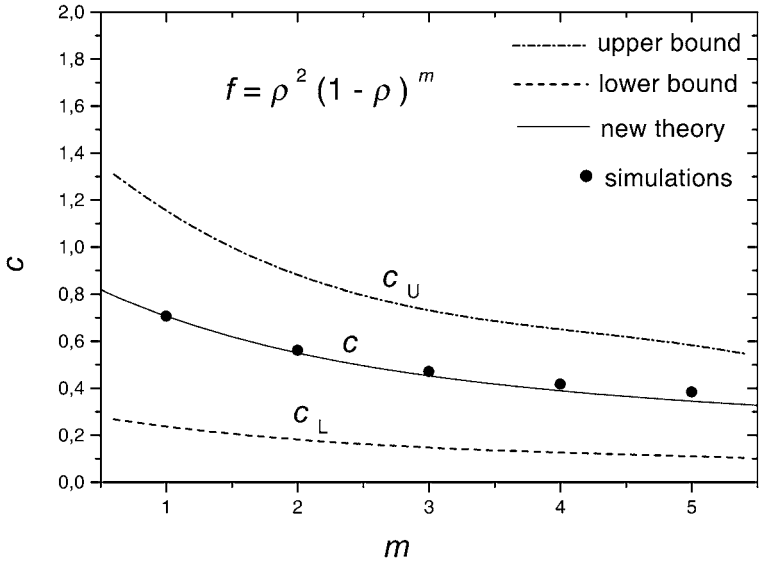

Fig. 2. Comparison between upper and lower bounds, numerical simulations and the prediction from our approximate method for chemical kinetics with $f(\rho) = \rho^2(1-\rho)^m$.

value of m is. This could have been expected since for the case considered ($n = 2$), the approximation (21) is less precise the larger the value of m is.

4.3. Time-Delayed Reaction-Diffusion

In many biological and physical applications, such as population invasions[3] and flame propagation,[1] the following time-delayed reaction-diffusion equation arises[14, 15, 18]

$$a\,\partial_{tt}\rho + \partial_t \rho = \partial_{xx}\rho + f(\rho) + af'(\rho)\,\partial_t \rho \tag{23}$$

where $a < 1$ is the dimensionless delay time. Let us here consider for definiteness, e.g., the case

$$f = \rho^2(1-\rho) \tag{24}$$

This source function is chosen for illustrative purposes, and it is easy to perform the analysis below for other forms of the source function. It has been already analyzed by means of variational methods in ref. 15, where lower and upper bounds for the front speed have been obtained. On the other hand, an approximated method, analogous to that developed in

Section 2 in the present paper, has been recently constructed[21] for the time-delayed case [Eq. (23)] instead of the classical case [Eq. (1)]. It is very easy to see that the approach in ref. 21 yields for the case corresponding to Eq. (24)

$$c = s/\sqrt{1+as^2} \tag{25}$$

where

$$s = \max_b \left\{ 2\sqrt{b}\, \frac{\int_0^1 g\rho[1-a(2\rho-3\rho^2)]\, d\rho}{\int_0^1 g[1-a(2\rho-3\rho^2)]\, d\rho} \right\} \tag{26}$$

and

$$g = (1-\rho)^{b(1+a)^2} \exp\left[\frac{9}{4} a^2 b\rho^4 - a^2 b\rho^3 + \frac{a^2+6a}{2} b\rho^2 + a(a+2)\, b\rho \right] \tag{27}$$

The validity of the methods in refs. 15 and 21 for different values of the delay parameter a has not been analyzed. This will be done below, but let us first use the Renormalization Group (RG) technique so that we can thereafter compare the advantages and drawbacks of the different methods. The RG technique has been never applied previously to time-delayed equations. In order to do so, we will treat the reduced delay a as a smallness parameter, so that it introduces a small perturbation to the usual, nondelayed equation (1). Thus, in the present paper we are treating the hyperbolic reaction-diffusion equation (23) as a perturbation of the parabolic equation (1). This is indeed reasonable, because Eq. (23) is a first-order approximation to the full time-delayed equation[18, 3] and, on the other hand, the RG technique is nothing but a first-order approach to the problem $\partial_t \rho = N(\rho)$, with $N(\rho)$ an arbitrary differential operator, which yields[19, 20]

$$c \simeq c_0 + \delta c \tag{28}$$

where[19, 20]

$$\delta c = -\frac{\int_{-\infty}^{+\infty} dz\, \omega(z) \dfrac{d\rho_0}{dz}\, \delta N\{\rho_0\}}{\int_{-\infty}^{+\infty} dz\, \omega(z) \left(\dfrac{d\rho_0}{dz} \right)^2} \tag{29}$$

with $\omega(z)$ an appropriate weight function (as explained below), and c_0 the speed of the unperturbed front $\rho_0(z)$, which in our case satisfies from Eqs. (23) and (24) for $a=0$,

$$\partial_t \rho_0 = \partial_{xx}\rho_0 + \rho_0^2(1-\rho_0)$$

It is easily seen that this equation has a solution with the profile

$$\rho_0(z) = \frac{1}{1+e^{z/\sqrt{2}}}$$

where $z = x - c_0 t$ and $c_0 = 1/\sqrt{2}$. The perturbation operator is easily found,

$$\begin{aligned}\delta N\{\rho_0\} &\equiv N\{\rho_0\} - N_{a=0}\{\rho_0\} \\ &= a[f'(\rho_0)\,\partial_t\rho_0 - \partial_{tt}\rho_0] \\ &= a\left[-c_0(2-3\rho_0)\,\rho_0\,\frac{d\rho_0}{dz} - c_0^2\,\frac{d^2\rho_0}{dz^2}\right]\end{aligned}$$

Then, by choosing the weight function $\omega(\rho) = e^{z/\sqrt{2}}$ we ensure the convergence of the integrals, and Eqs. (28)–(29) yield after integration

$$c_{\mathrm{RG}} \simeq \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{40}\,a + O(a^2) \tag{30}$$

In Fig. 3 we compare the results of numerical simulations of Eq. (23) to those from the approximate method (Eq. (25)), from the RG approach (Eq. (30)), and also with the lower and upper bounds known from a variational principle.[15] The agreement of the approximate method with the numerical simulations is clear again. The speed decreases for increasing delay time, as was to be expected intuitively. It is not surprising that the RG approach becomes less reliable at higher delays, whereas the approximate method does not, since the RG approach is a first-order approximation valid only for small enough delays [see Eq. (30)]. However, since in most applications $a < 1$,[22] the RG technique is rather accurate even for relatively high delays (see Fig. 3), which makes it appealing because it gives a simple result [Eq. (30)] that is much more easy and practical to handle than that of our approximate method [Eqs. (25)–(27)]. Moreover, as mentioned above, the derivations of Eq. (23) show that it is in fact valid for low values of a only.[18, 3] It means that a very precise determination of the speed, such as that given by Eqs. (25)–(27), will not hold for high values

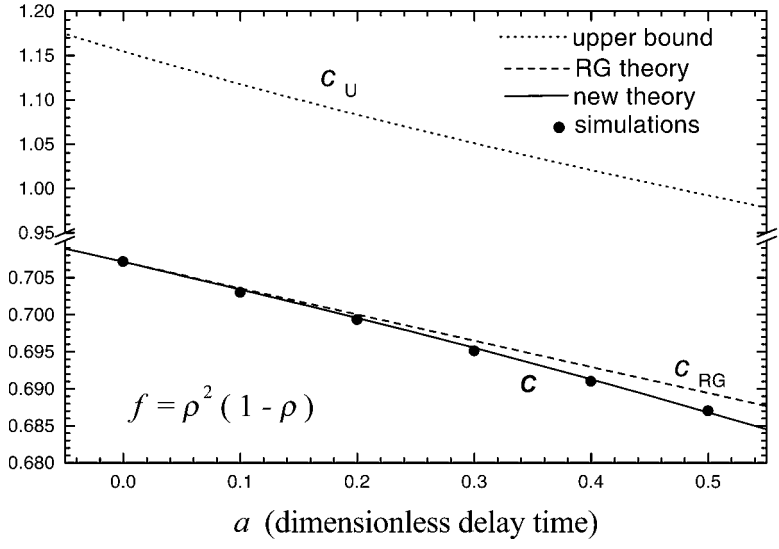

Fig. 3. Time-delayed diffusion. Dots are from numerical simulations. The full line is the approximate prediction (25), whereas the dashed one is our new RG prediction (30). It is seen that both methods remain accurate for rather high delays. The dotted line is the upper bound. The lower bound (not shown) yields $c > 0.46$.

of a because the corresponding differential equation (23) breaks down itself. This again lead us to argue that our new result (30) is more interesting in practice than the alternative result (25). However, this conclusion holds only for time-delayed equations. For nondelayed reaction-diffusion, the RG approach cannot be applied in most cases, because of the abscence of an smallness parameter in general. Then, the approximate method presented in Section 2 is very useful (see Figs. 1 and 2).

5. CONCLUSIONS

We have dealt with the problem of the determination of the speed of wavefront solutions to reaction-diffusion equations. We have applied an approximate method, presented in Section 2, to a variety of source terms of practical interest. In Section 3, we have shown that our approach makes it possible to derive some analytical results that were previously known from other methods. The examples in Section 3 are also very useful from a more practical perspective, since they help to get used to the way in which calculations and approximations are carried out within our method, before one tries to apply it to more complicated cases (such as those in Section 3). However, it is also important that Section 3 shows very clearly how our

technique makes an exact solution $\rho(z)$ unnecessary. In Section 4 we have tackled several problems for which analytical results are not known, so our method has been checked by means of numerical simulations. In Section 4.1, we have considered a source function which has been used in forest fire research, showing that our method is more accurate than previous ones, which had made it possible to find only upper and lower bounds for the speed (Fig. 1). In Section 4.2, we have considered a chemical kinetics model, which has been used in microbiology, and for which our method again improves previously-known bounds (Fig. 2). Finally, in Section 4.3 the renormalization group (RG) approach has been applied for the first time to the case of time-delayed equations, and its results have been compared to those of the time-delayed analog to our approximate method (Fig. 3). Obviously, the RG approach has not been applied to the examples in Sections 4.1 and 4.2 because in general no smallness parameter (analog to the delay a in Section 4.3) appears in those cases. It is very interesting, though, that for time-delayed equations our new alternative approach (based on the RG technique) is likely to become more useful in practice than the approximate variational approach, in spite of being less accurate for high values of the delay time.

Our work can also be useful, by proceeding analogously, to determine the speed of reaction-diffusion fronts for other source terms and reaction-diffusion equations of physical or biological importance.

Variational principles have been succesfully applied to the front speed problem for some time. Chen's trial functions made the Hadeler–Rothe principle practically usable (see Section 6 in the second paper cited under ref. 20). The Benguria–Depassier (BD) principle[4] was another major advance in this line of research. Our procedure is based on the BD theory; however, it goes further by making an additional approximation which allows to determine the trial function g explicitly; in contrast, in previous work one had to choose several functions g until acceptable (and often less accurate) bounds on the speed were derived.

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